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Metric Entropy of Subsets of Absolutely Convergent Fourier Series

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In 1967, G. G. Lorentz wrote in his paper [Lor, p. 922] "An interesting problem is to derive, from the behaviour of $\mathscr{H}_\varepsilon(K)$, properties of the Fourier coefficients of some, or of most functions $f \in K$, $K \subset L^p$ or $K \subset C$. To give only one example: Is it true, for a subset K of C, that $\mathscr{H}_\varepsilon(K) \geqslant \mathrm{Const}(1/\varepsilon)^2$ implies the existence of a function $f \in K$ whose Fourier series is not absolutely convergent?"

The easy answer to this particular question is "No". Indeed, take K consisting of functions $\alpha_m e^{imx}$, $m = 1, 2, ...$ where α_m is a sequence of numbers which decreases slowly enough to zero. This class is obviously compact, and all functions are well separated, hence its entropy could be very big.

Nevertheless it seems interesting to estimate the entropy of compact subsets of the set A of continuous functions with absolutely convergent Fourier series.

Let us recall the definitions. The class H_α is the set of real or complex-valued, 2π-periodic functions f such that $\hat{f}(0) = 0$ and $\|\Delta_h^l f\|_\infty \leqslant |h|^\alpha$ where the difference of integer order $l > \alpha$ is taken with the step h.

We denote by A_p, $0 < p < 2$ the class of functions $f(x)$ such that the sequence of the Fourier coefficients belongs to the space l^p, and such that

$$\|f\|_{A_p} = \left\{\sum_\nu |\hat{f}(\nu)|^p\right\}^{1/p} \leqslant 1.$$

For the notion of ε-entropy see for example [Ti, p. 274] or [Pis, Ch. 5]. Let K be a compact set in a Banach space X. The ε-entropy $\mathscr{H}_\varepsilon(K; X)$ (or

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simply $\mathscr{H}(K, \varepsilon)$) is the logarithm to the base two of the number of points in the minimal ε-net for K. We use also the inverse characteristic, so-called entropy numbers given by

$$e_m(K; X) = \inf \left\{ \varepsilon \colon \exists x_1, ..., x_{2^m} \colon K \subset \bigcup_{j=0}^{2^m} (x_j + \varepsilon B_X) \right\},$$

where the infimum is taken over all ε such that K can be covered by 2^m open balls εB_X of radius ε (we denote by B_X the open unit ball of the Banach space X).

We write $a_n \ll b_n$ if there exists constant C, independent of n such that $a_n \leqslant C b_n$.

We start with the result for the classical Wiener algebra of absolutely convergent Fourier series $A = A_1$.

THEOREM 1. *The following estimates hold*

$$\mathscr{H}_\varepsilon(A \cap H_\alpha; L^\infty) \simeq \frac{1}{\varepsilon^2}, \qquad \alpha = 1/2$$

$$\frac{\log(1/\varepsilon)}{\varepsilon^2} \ll \mathscr{H}_\varepsilon(A \cap H_\alpha; L^\infty) \ll \frac{\log^2(1/\varepsilon)}{\varepsilon^2}, \qquad 0 < \alpha < 1/2.$$

Remark. The case $\alpha > 1/2$ is not interesting because by the classical Bernstein theorem (see, for example [Zy, Ch. 6, Th. 3.1]) we have the embedding $H_\alpha \subset A$ and the order of the entropy of this class $\mathscr{H}_\varepsilon(H_\alpha; L^\infty) \simeq \varepsilon^{-1/\alpha}$ is well known [KoT], (see also [Lor. p. 920]).

Remark. If $H_\alpha \cap A_I$ denotes the class H_α functions which coincide with a function from A on an interval I, $|I| < 2\pi$ then

$$\mathscr{H}_\varepsilon(H_\alpha \cap A_I; L^\infty) \simeq \varepsilon^{-1/\alpha}, \qquad 0 < \alpha < 1/2.$$

Indeed, the above estimate follows from the embedding $H_\alpha \cap A_I \subset H_\alpha$. In order to construct the set of ε-separated functions for the estimate from below, we can take each function from the corresponding set on $[0, 2\pi] \backslash I$, and extend each function by zero onto I. ∎

Proof of the theorem. Let $\alpha = 1/2$. The estimate from above follows directly from the obvious embedding $A \cap H_{1/2} \subset H_{1/2}$ and the classical result (see the reference above). We prove the estimate from below. For this we construct a big enough set of ε-separated functions. Let $\varepsilon = \sqrt{\pi/n}$, and consider the function

$$\lambda_h(x) = \begin{cases} 0, & |x| > h \\ 1, & x = 0 \\ \text{linear}, & 0 < |x| < h \end{cases}$$

extended periodically outside of $[-\pi, \pi]$. Take $h = \pi/n$. Then the function $\sqrt{h}\, \lambda_h(x)$ belongs to $H_{1/2}$. We consider the functions

$$F_t(x) = \sqrt{h} \sum_{k=0}^{n} r_k(t)\, \lambda_h(x - (2k+1)\, h),$$

where $r_k(t) = \operatorname{sgn} \sin 2^k \pi t$ are the Rademacher functions. For each t, the function $F_t(x) \in H_{1/2}$, and if $|t_1 - t_2| > \frac{\pi}{2^n}$ then

$$\|F_{t_1} - F_{t_2}\|_\infty \gg \sqrt{h}.$$

Therefore, we have a set of 2^n ε-separated functions. We have to check only that each one is in the space A.

The Fourier series of $F_t(x)$ is

$$\begin{aligned} F_t(x) &= \sum_{k=0}^{n} h^{3/2} \left[\frac{1}{2} + \sum_{\nu=1}^{\infty} \left(\frac{\sin \nu h}{\nu h} \right)^2 r_k(t) \cos \nu(x + (2k+1)\, h) \right] \\ &= h^{3/2} \left[\frac{1}{2} + \sum_{\nu=1}^{\infty} \left(\frac{\sin \nu h}{\nu h} \right)^2 \left(\sum_{k=0}^{n} r_k(t) \cos \nu(x + (2k+1)\, h) \right) \right]. \end{aligned}$$

Hence

$$\|F_t\|_A \leqslant h^{3/2} \sum_{\nu=0}^{\infty} \left(\frac{\sin \nu h}{\nu h} \right)^2 \left| \sum_{k=0}^{n} r_k(t) \cos \nu(x + (2k+1)\, h) \right|.$$

Therefore by the Chebyshev inequality

$$\begin{aligned} m\{t \colon \|F_t\|_A > y\} \leqslant \frac{1}{y} \int_0^1 h^{3/2} \sum_{\nu=0}^{\infty} \left(\frac{\sin \nu h}{\nu h} \right)^2 & \\ \times \left| \sum_{k=0}^{n} r_k(t) \cos \nu(x + (2k+1)\, h) \right| dt. & \end{aligned}$$

The orthogonality of the Rademacher functions, and two simple inequalities $\sin x \leqslant x$ and $|\sin x| \leqslant 1$ give us the estimate

$$m\{t: \|F_t\|_A > y\} \leqslant \frac{\text{Const}}{y}.$$

If we choose y such that the right-hand part $\leqslant 1/2$ then for at least half of the constructed functions we have

$$\|F_t\|_A < y.$$

The estimate is proved.

Remark. The same estimate from below can be obtained from the following simple observation. The set $A \cap H_{1/2}$ contains the set of all trigonometric polynomials $T_n(x) = \sum_{|k| \leqslant n} c_k e^{ikx}$ such that $\|T_n\|_\infty \leqslant n^{-1/2}$. Hence, considering the $(2n+1)$-dimensional subspace of L_∞ with the coefficients c_k as coordinates, covering this subset of it with the balls by standard way, comparing the Euclidean volumes in this subspace, and choosing the optimal n we obtain the estimate.

Let us consider the case $0 < \alpha < 1/2$. Take $\varepsilon = n^{-1/2}$. The estimate from above is an easy consequence of the following three facts.

(a) Classical approximation by Fejer means of the Fourier series (see, for example [Zy, Ch. 3, Th. 3.15])

$$\|f(x) - F_n(f; x)\|_\infty \ll n^{-\alpha}.$$

We take Fejer means of order $m = n^{1/2\alpha}$, reducing the problem to the estimate of the entropy of the trigonometric polynomials of order $\leqslant 2m$.

(b) The best approximation by trigonometric polynomials with a prescribed *number* of harmonics:

LEMMA [DvT]. *For every trigonometric polynomial $T_m = \sum_{k=-m}^{m} c_k e^{ikx}$ there exists a trigonometric polynomial t_n with n harmonics such that*

$$\|T_m - t_n\|_\infty \leqslant \begin{cases} n^{1/2 - 1/p} \sqrt{\lg (m/n)}\, \|T_m\|_{A_p} & 0 < p \leqslant 1 \\ m^{1-1/p} n^{-1/2} \sqrt{\lg (m/n)}\, \|T_m\|_{A_p} & 1 < p \leqslant 2. \end{cases}$$

This reduces the problem to the estimate of the entropy of the unit ball in the space of trigonometric polynomials with n harmonics.

(c) Estimate of ε-entropy of the unit ball of finite dimensional space.

LEMMA [KoT]. *See also [Pi, p. 63] If X is an n-dimensional space, then*

$$n \log \frac{1}{\varepsilon} \leqslant \mathscr{H}_\varepsilon(B_X; X) \leqslant n \log\left(1+\frac{2}{\varepsilon}\right).$$

Finally we apply this estimate to the unit ball of each subspace and count the possible number of subspaces, which is at most $\binom{m}{n}$.

Let us prove the estimate from below. With the same $\varepsilon = n^{-1/2}$ we construct the ε-distinguished set of functions by a linear combination of shifts of the $\lambda_h(x)$. Take $h = \pi/n^{1/2\alpha}$. Divide $[0, 2\pi]$ into $n^{1/2\alpha}+1$ equal intervals by $n^{1/2\alpha}$ points x_j. Then the function $h^\alpha \lambda_h(x) \in H_\alpha$. Let

$$F_t = \sum_{j=1}^{n} h^\alpha r_j(t)\, \lambda_h(x - x_{n_j}),$$

where n points x_{n_j} are chosen randomly among all $n^{1/2\alpha}$ knots. Then

$$\|F_t\|_A \ll \frac{2\pi}{n^{1/2}} \left[h + h \sum_{v=1}^{\infty} \left(\frac{\sin vh}{vh}\right)^2 \left| \sum_{j=0}^{n} r_j(t) \cos v x_j \right| \right].$$

We choose signs (value t) such that $\|F_t\|_A \leqslant \text{Const}$. It is obvious that for two different functions

$$\|F_t - F'_t\|_\infty \geqslant n^{-1/2}.$$

The total number of functions is $\binom{n^{1/2\alpha}}{n}$, which can be estimated from below by $n^{(1/2\alpha - 1)n}$. This gives the estimate from below.

Let us denote by H^q_α the class of functions f such that $\hat{f}(0) = 0$ and $\|\Delta^l_h f\|_q \leqslant |h|^\alpha$, where the difference of integer order $l > \alpha$ is taken with the step h.

COROLLARY. *Suppose $2 < q < \infty$. Then*

$$\frac{\log(1/\varepsilon)}{\varepsilon^2} \ll \mathscr{H}_\varepsilon(A \cap H^q_\alpha; L^\infty) \ll \frac{\log^2(1/\varepsilon)}{\varepsilon^2}, \qquad \frac{1}{q} < \alpha \leqslant \frac{1}{2}.$$

Proof. Indeed, the result follows directly from the embeddings $H_\alpha \subset H^q_\alpha \subset H_{\alpha - 1/q}$ (the left-hand part is obvious, for the right-hand part see, for example [Nik, Ch. 6]) and Theorem 1.

The order of the entropy of the subsets of A_p, $0 < p < 1$ is also independent of α. We formulate the result in terms of the entropy numbers.

THEOREM 2. *The following estimates hold*

$$e_m(A_p \cap H_\alpha; L^\infty) \simeq \frac{1}{m^\alpha}, \qquad \alpha = \frac{1}{p} - \frac{1}{2}$$

$$\left(\frac{\log m}{m}\right)^{1/p-1/2} \ll e_m(A_p \cap H_\alpha; L^\infty)$$

$$\ll \left(\frac{\log m}{m}\right)^{1/p-1/2} \sqrt{\log m}, \qquad 0 < \alpha < \frac{1}{p} - \frac{1}{2}.$$

The proof of the above estimate is the same as in Theorem 1. To prove the estimate from below, the modified function $\bar{\lambda}_h = \lambda_h * \cdots * \lambda_h$ is needed. The convolution is taken a number of times sufficient for the necessary smoothness of the function $\bar{\lambda}_h$.

Another fact which has to be taken into account is that

$$\|\lambda_h\|_{A_p} \simeq h^{1-(1/p)}.$$

This can be checked by simple calculation.

But the case of A_p when $1 < p < 2$ is completely different, here the order of entropy depends on the smoothness of the functions.

THEOREM 3. *The following estimates hold*

$$e_m(A_p \cap H_\alpha; L^\infty) \simeq \frac{1}{m^\alpha}, \qquad \alpha = \frac{1}{p} - \frac{1}{2}$$

$$\left(\frac{\log m}{m}\right)^{(\alpha/2)/(\alpha+1/p')} \ll e_m(A_p \cap H_\alpha; L^\infty)$$

$$\ll \left(\frac{\log m}{m}\right)^{(\alpha/2)/(\alpha+1/p')} \sqrt{\log m},$$

$$0 < \alpha < \frac{1}{p} - \frac{1}{2},$$

where as usual $1/p + 1/p' = 1$.

The proof of this statement follows the same scheme, but the second line of the [DvT] Lemma has to be applied for the above estimates.

The same proof can be used for the classes of functions of small smoothness. Let H_ω^α be the class of functions f such that $\|f(x) - f(x+h)\|_\infty \leqslant (\log 1/h)^{-\alpha}$ for fixed $0 < \alpha < \infty$ and arbitrary $0 < h < 2\pi$.

THEOREM 4. *The following estimates hold*

$$m^{-\alpha/2\alpha+1} \ll e_m(H_\omega^\alpha \cap A; L_\infty) \ll m^{-\alpha/2\alpha+2}.$$

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